XIV. *Analytical Researches connected with* STEINER's *Extension of* MALFATTI's *Problem. By* ARTHUR CAYLEY, *M.A., Fellow of Trinity College, Cambridge. Communicated by* J. J. SYLVESTER, *Esq., F.R.S.*

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THE problem, in a triangle to describe three circles each of them touching the two others and also two sides of the triangle, has been termed after the Italian geometer by whom it was proposed and solved, MALFATTI's problem. The problem which I refer to as STEINER's extension of MALFATTI's problem is as follows:—"To determine three sections of a surface of the second order, each of them touching the two others, and also two of three given sections of the surface of the second order," a problem proposed in STEINER's memoir, 'Einige geometrische Betrachtungen,' Crelle, t. i. The geometrical construction of the problem in question is readily deduced from that given in the memoir just mentioned for a somewhat less general problem, viz. that in which the surface of the second order is replaced by a sphere; it is for the sake of the analytical developments to which the problem gives rise, that I propose to resume here the discussion of the problem. The following is an analysis of the present memoir:—

§ 1. Contains a lemma which appears to me to constitute the foundation of the analytical theory of the sections of a surface of the second order.

§ 2. Contains a statement of the geometrical construction of STEINER's extension of MALFATTI's problem.

§ 3. Is a verification, founded on a particular choice of coordinates, of the construction in question.

§ 4. In this section, referring the surface of the second order to absolutely general coordinates, and after an incidental solution of the problem to determine a section touching three given sections, I obtain the equations for the solution of STEINER's extension of MALFATTI's problem.

§ 5. Contains a separate discussion of a system of equations, including as a particular case the equations obtained in the preceding section.

§§ 6 and 7. Contain the application of the formulæ for the general system to the equations in § 4, and the development and completion of the solution.

§ 8. Is an extension of some preceding formulæ to quadratic functions of any number of variables.

§ 1. *Lemma relating to the sections of a surface of the second order.*

If

$$ax^2+by^2+cz^2+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw=0$$

be the equation of a surface of the second order, and

$$\mathfrak{A}x^2+\mathfrak{B}y^2+\mathfrak{C}z^2+\mathfrak{D}w^2+2\mathfrak{F}yz+2\mathfrak{G}zx+2\mathfrak{H}xy+2\mathfrak{L}xw+2\mathfrak{M}yw+2\mathfrak{N}zw=0$$

the reciprocal equation, the condition that the two sections

$$\lambda x+\mu y+\nu z+\varrho w=0$$
$$\lambda' x+\mu' y+\nu' z+\varrho' w=0$$

may touch, is

$$(\mathfrak{A}\lambda^2+\mathfrak{B}\mu^2+\mathfrak{C}\nu^2+\mathfrak{D}\varrho^2+2\mathfrak{F}\mu\nu+2\mathfrak{G}\nu\lambda+2\mathfrak{H}\lambda\mu+2\mathfrak{L}\lambda\varrho+2\mathfrak{M}\mu\varrho+2\mathfrak{N}\nu\varrho)^{\frac{1}{2}}$$
$$\times(\mathfrak{A}\lambda'^2+\mathfrak{B}\mu'^2+\mathfrak{C}\nu'^2+\mathfrak{D}\varrho'^2+2\mathfrak{F}\mu'\nu'+2\mathfrak{G}\nu'\lambda'+2\mathfrak{H}\lambda'\mu'+2\mathfrak{L}\lambda'\varrho'+2\mathfrak{M}\mu'\varrho'+2\mathfrak{N}\nu'\varrho')^{\frac{1}{2}}.$$
$$=\Big(\mathfrak{A}\lambda\lambda'+\mathfrak{B}\mu\mu'+\mathfrak{C}\nu\nu'+\mathfrak{D}\varrho\varrho'+\mathfrak{F}(\mu\nu'+\mu'\nu)+\mathfrak{G}(\nu\lambda'+\nu'\lambda)+\mathfrak{H}(\lambda\mu'+\lambda'\mu)+\mathfrak{L}(\lambda\varrho'+\lambda'\varrho)$$
$$+\mathfrak{M}(\mu\varrho'+\mu'\varrho)+\mathfrak{N}(\nu\varrho'+\nu'\varrho)\Big).$$

And in particular if the equation of the surface be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+pw^2=0,$$

the condition of contact is

$$\left(\mathfrak{A}\lambda^2+\mathfrak{B}\mu^2+\mathfrak{C}\nu^2+2\mathfrak{F}\mu\nu+2\mathfrak{G}\nu\lambda+2\mathfrak{L}\lambda\mu+\frac{\mathrm{K}}{p}\varrho^2\right)^{\frac{1}{2}}$$
$$\left(\mathfrak{A}\lambda'^2+\mathfrak{B}\mu'^2+\mathfrak{C}\nu'^2+2\mathfrak{F}\mu'\nu'+2\mathfrak{G}\nu'\lambda'+2\mathfrak{H}\lambda'\mu'+\frac{\mathrm{K}}{p}\varrho'^2\right)^{\frac{1}{2}}$$
$$=\left(\mathfrak{A}\lambda\lambda'+\mathfrak{B}\mu\mu'+\mathfrak{C}\nu\nu'+\mathfrak{F}(\mu\nu'+\mu'\nu)+\mathfrak{G}(\nu\lambda'+\nu'\lambda)+\mathfrak{H}(\lambda\mu'+\lambda'\mu)+\frac{\mathrm{K}}{p}\varrho\varrho'\right),$$

in which last formula

$$\mathfrak{A}=bc-f^2,\quad \mathfrak{B}=ca-g^2,\quad \mathfrak{C}=ab-h^2,$$
$$\mathfrak{F}=gh-af,\quad \mathfrak{G}=hf-bg,\quad \mathfrak{H}=fg-ch,$$
$$\mathrm{K}=abc-af^2-bg^2-ch^2+2fgh.$$

§ 2.

In order to state in the most simple form the geometrical construction for the solution of STEINER'S extension of MALFATTI'S problem, let the given sections be called for conciseness the determinators*; any two of these sections lie in two different cones, the vertices of which determine with the line of intersection of the planes of the determinators, two planes which may be termed bisectors; the six bisectors pass three and three through four straight lines; and it will be convenient to use the term bisectors to denote, not the entire system, but any three bisectors passing through the same line. Consider three sections, which may be termed tactors, each of them touching a determinator and two bisectors, and three other sections (which may be termed separators) each of them passing through the point of contact

* I use the words 'determinators,' &c. to denote indifferently the sections or the planes of the sections; the context is always sufficient to prevent ambiguity.

of a determinator and tactor and touching the other two tactors; the separators will intersect in a line which passes through the point of intersection of the determinators. The three required sections, or as I shall term them the resultors, are determined by the conditions that each resultor touches two determinators and two separators, the possibility of the construction being implied as a theorem. The *à posteriori* verification may be obtained as follows:—

§ 3.

Let $x=0$, $y=0$, $z=0$ be the equations of the resultors, $w=0$ the equation of the polar of the point of intersection of the resultors. Since the resultors touch two and two, the equation of the surface is easily seen to be of the form

$$2yz+2zx+2xy+w^2=0\,*.$$

The determinators are sections each of them touching two resultors, but otherwise arbitrary; their equations are

$$-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0$$

$$\frac{1}{2\beta}x-\beta y+\frac{1}{2\beta}z+w=0$$

$$\frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z+w=0.$$

The separators are sections each of them touching two resultors at their point of contact (or what is the same thing, passing through the line of intersection of two resultors), and all of them having a line in common. Their equations may be taken to be

$$cy-bz=0,\quad az-cx=0,\quad bx-ay=0,$$

the values of a, b, c remaining to be determined. Now before fixing the values of these quantities, we may find three sections each of them touching a determinator at a point of intersection with the section which corresponds to it of the sections $cy-bz=0$, $az-cx=0$, $bx-ay=0$, and touching the other two of the last-mentioned sections; and when a, b, c have their proper values the sections so found are the tactors. For, let $\lambda x+\mu y+\nu z+\varrho w=0$ be the equation of a section touching the determinator $-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0$, and the two sections $bx-ay=0$, $az-cx=0$, and suppose

$$\Delta^2=\lambda^2+\mu^2+\nu^2-2\mu\nu-2\nu\varrho-2\lambda\mu-2\varrho^2,$$

the conditions of contact with the sections $bx-ay=0$, $az-cx=0$ are found to be

$$(b+a)\Delta=(b+a)\lambda-(b+a)\mu-(b-a)\nu$$
$$(c+a)\Delta=(c+a)\lambda-(c-a)\mu-(c+a)\nu,$$

values, however, which suppose a correspondence in the signs of the radicals. Thence

* The reciprocal form is, it should be noted,

$$x^2+y^2+z^2-2yz-2zx-2xy-2w^2=0.$$

$(b+a)\mu=(c+a)\nu$; or since the ratios only of the quantities $\lambda, \mu, \nu, \varrho$ are material, $\mu=c+a$, $\nu=b+a$, and therefore

$$\Delta^2=\lambda^2-2(2a+b+c)\lambda+(b-c)^2-2\varrho^2=(\lambda-b-c)^2,$$

or
$$\varrho^2=-2(a\lambda+bc).$$

Whence the equation to a section touching $bx-ay=0$, $az-cx=0$ is

$$\lambda x+(c+a)y+(b+a)z+\sqrt{-2(a\lambda+bc)}\,w=0.$$

And to express that this touches the determinator in question, we have

$$\pm\alpha(\lambda-b-c)=\left(\alpha+\frac{1}{\alpha}\right)\lambda-\alpha(2a+b+c)+2\sqrt{-2(a\lambda+bc)}\,;$$

and selecting the upper sign,

$$\frac{1}{\alpha}\lambda-2a\alpha=-2\sqrt{-2(a\lambda+bc)}\,;$$

whence

$$\lambda=-2\alpha(a\alpha-\sqrt{-2bc}),\qquad \sqrt{-2(a\lambda+bc)}=(2a\alpha-\sqrt{-2bc})\,;$$

or the section touching the determinator and the sections $bx-ay=0$, $az-cx=0$ is

$$-2\alpha(a\alpha-\sqrt{-2bc})x+(c+a)y+(b+a)z+(2a\alpha-\sqrt{-2bc})w=0\,;$$

and at the point of contact with the determinator

$$-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0$$

$$2yz+2zx+2xy+w^2=0.$$

Eliminating w between the first and second equations and between the second and third equations,

$$\sqrt{-2bc}\left(\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z\right)+cy+bz=0,$$

$$\left(\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z\right)^2+2yz=0\,;$$

and from these equations $(cy-bz)^2=0$, or the point of contact lies in the section $cy-bz=0$. It follows that the equations of the tactors are

$$\begin{aligned}
&-2\alpha(a\alpha-\sqrt{-2bc})x+(c+a)y+(b+a)z+(2a\alpha-\sqrt{-2bc})w=0\\
&(c+b)x-2\beta(b\beta-\sqrt{-2ca})y+(a+b)z+(2b\beta-\sqrt{-2ca})w=0\\
&(b+c)x+(a+c)y-2\gamma(c\gamma-\sqrt{-2ab})z+(2c\gamma-\sqrt{-2ab})w=0,
\end{aligned}$$

where a, b, c still remain to be determined.

Now the separators pass through the point of intersection of the determinators; the equations of these give for the point in question,

$$\begin{aligned}
x:y:z:w=&(2\beta\gamma+1)(-\alpha+\beta+\gamma+2\alpha\beta\gamma)\\
:\ &(2\gamma\alpha+1)(\alpha-\beta+\gamma+2\alpha\beta\gamma)\\
:\ &(2\alpha\beta+1)(\alpha+\beta-\gamma+2\alpha\beta\gamma)\\
:\ &4\alpha^2\beta^2\gamma^2-1+\alpha^2+\beta^2+\gamma^2\,;
\end{aligned}$$

and the values of a, b, c are therefore

$$\begin{aligned} a:b:c=&(2\beta\gamma+1)(-\alpha+\beta+\gamma+2\alpha\beta\gamma)\\ :&\ (2\gamma\alpha+1)(\alpha-\beta+\gamma+2\alpha\beta\gamma)\\ :&\ (2\alpha\beta+1)(\alpha+\beta-\gamma+2\alpha\beta\gamma),\end{aligned}$$

which are to be substituted for a, b, c in the equations of the separators and tactors respectively.

Now proceeding to find the bisectors, let $\lambda x+\mu y+\nu z+\varrho w=0$ be the equation of a section touching the determinators,

$$\frac{1}{2\beta}x-\beta y+\frac{1}{2\beta}z+w=0, \qquad \frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z+w=0.$$

And suppose, as before, $\Delta^2=\lambda^2+\mu^2+\nu^2-2\mu\nu-2\nu\lambda-2\lambda\mu-2\varrho^2$; the conditions of contact are

$$\pm\beta\Delta=\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\varrho$$

$$\mp\gamma\Delta=\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\varrho,$$

where it is necessary, for the present purpose, to give opposite signs to the radicals. For if the radicals had the same sign, it would follow that

$$\frac{1}{\beta}\left[\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\varrho\right]-\frac{1}{\gamma}\left[\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\varrho\right]=0\,;$$

or the equation $\lambda x+\mu y+\nu z+\varrho w=0$ would pass through the point

$$x:y:z:w=0\,;\ \frac{1}{\beta^2}\,;\ \frac{1}{\gamma^2}:-\frac{2}{\beta}+\frac{2}{\gamma}\,;$$

or the section would be a tangent section of the two determinators of the same class with the resultor $x=0$, which ought not to be the case. The proper formula is

$$\frac{1}{\beta}\left[\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\varrho\right]+\frac{1}{\gamma}\left[\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\varrho\right]=0.$$

And this equation being satisfied, the section

$$\lambda x+\mu y+\nu z+\varrho w=0$$

passes through a point

$$x:y:z:w=2:-\frac{1}{\beta^2}:-\frac{1}{\gamma^2}:-\frac{2}{\beta}-\frac{2}{\gamma}.$$

The bisector passes through this point and the line of intersection of the determinators; its equation is

$$\frac{1}{\beta}\left(\frac{1}{2\beta}x-\beta y+\frac{1}{2\beta}z+w\right)-\frac{1}{\gamma}\left(\frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z+w\right)=0\,;$$

or reducing and completing the system, the equations of the bisectors are

$$\left(\frac{1}{2\beta^2}-\frac{1}{2\gamma^2}\right)x-\left(1+\frac{1}{2\gamma^2}\right)y+\left(1+\frac{1}{2\beta^2}\right)z+\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)w=0,$$

$$\left(1+\frac{1}{2\gamma^2}\right)x+\left(\frac{1}{2\gamma^2}-\frac{1}{2\alpha^2}\right)y-\left(1+\frac{1}{2\alpha^2}\right)z+\left(\frac{1}{\gamma}-\frac{1}{\alpha}\right)w=0,$$

$$-\left(1+\frac{1}{2\beta^2}\right)x+\left(1+\frac{1}{2\alpha^2}\right)y+\left(\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}\right)z+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)w=0.$$

And in order to verify the geometrical construction, it only remains to show that each bisector touches two tactors. Consider the bisector and tactor

$$-\left(1+\frac{1}{2\beta^2}\right)x+\left(1+\frac{1}{2\alpha^2}\right)y+\left(\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}\right)z+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)w=0,$$

$$-2\alpha(a\alpha-\sqrt{-2bc})x+(c+a)y+(b+a)z+(2a\alpha-\sqrt{-2bc})w=0;$$

and represent these for a moment by

$$\lambda x+\mu y+\nu z+\varrho w=0,\quad \lambda' x+\mu' y+\nu' z+\varrho' w=0.$$

If Δ be the same as before, and Δ' the like function of λ', μ', ν', ϱ', also if

$$\Phi=\lambda\lambda'+\mu\mu'+\nu\nu'-(\mu\nu'+\mu'\nu)-(\nu\lambda'+\nu'\lambda)-(\lambda\mu'+\lambda'\mu)-2\varrho\varrho',$$

then

$$\Delta^2=\left(2+\frac{1}{\alpha\beta}\right)^2,$$

$$\Delta'^2=(2a\alpha^2-2\alpha\sqrt{-2bc}+b+c)^2,$$

$$\Phi=a\alpha^2\left(2+\frac{1}{\alpha\beta}\right)^2-2\alpha\sqrt{-2bc}\left(2+\frac{1}{\alpha\beta}\right)+c\left(2+\frac{1}{\beta^2}\right);$$

and the condition of contact $\Delta\Delta'=\Phi$ (taking the proper sign for the radicals) becomes

$$\left(2+\frac{1}{\alpha\beta}\right)(2a\alpha^2-2\alpha\sqrt{-2bc}+b+c)=a\alpha^2\left(2+\frac{1}{\alpha\beta}\right)^2-2\alpha\sqrt{-2bc}\left(2+\frac{1}{\alpha\beta}\right)+c\left(2+\frac{1}{\beta^2}\right);$$

or reducing,

$$a\alpha-b\beta+c\frac{\alpha-\beta}{2\alpha\beta+1}=0,$$

an equation which is evidently not altered by the interchange of a, α and b, β. The conditions, in order that each bisector may touch two tactors, reduce themselves to the three equations,

$$a\alpha-b\beta+c\frac{\alpha-\beta}{2\alpha\beta+1}=0,$$

$$a\frac{\beta-\gamma}{2\beta\gamma+1}+b\beta-c\gamma=0,$$

$$-a\alpha+b\frac{\gamma-\alpha}{2\alpha\gamma+1}+c\gamma=0,$$

which are satisfied by the values above found for the quantities a, b, c. The possibility and truth of the geometrical construction are thus demonstrated.

§ 4.

Let it be in the first instance proposed to find the equation of a section touching all or any of the sections $x=0$, $y=0$, $z=0$ of the surface of the second order,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+pw^2=0.$$

Any section whatever of this surface may be written in the form

$$(a\lambda+h\mu+g\nu)x+(h\lambda+b\mu+f\nu)y+(g\lambda+f\mu+c\nu)z+\sqrt{-p}\nabla w=0,$$

where

$$\nabla^2 = a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu - \mathrm{K},$$

where λ, μ, ν are indeterminate. And considering any other section represented by a like equation,

$$(a\lambda' + h\mu' + g\nu')x + (h\lambda' + b\mu' + f\nu')y + (g\lambda' + f\mu' + c\nu')z + \sqrt{-p}\nabla' w = 0,$$

where

$$\nabla' = a\lambda'^2 + b\mu'^2 + c\nu'^2 + 2f\mu'\nu' + 2g\nu'\lambda' + 2h\lambda'\mu' - \mathrm{K},$$

it may be shown by means of the lemma previously given, that the condition of contact is

$$a\lambda\lambda' + b\mu\mu' + c\nu\nu' + f(\mu\nu' + \mu'\nu) + g(\nu\lambda' + \nu'\lambda) + h(\lambda\mu' + \lambda'\mu) \pm \mathrm{K} = \nabla\nabla'.$$

Suppose that λ', μ', ν' satisfy the equations

$$\nabla' = 0,$$
$$h\lambda' + b\mu' + f\nu' = 0,$$
$$g\lambda' + f\mu' + c\nu' = 0,$$

so that the last mentioned section becomes $x=0$; and observing that the first of the above equations may be transformed into

$$a\lambda' + h\mu' + g\nu' = \frac{\mathrm{K}}{\lambda'},$$

it is easy to obtain $\lambda' = \sqrt{\mathfrak{A}}$, $\mu' = \frac{\mathfrak{H}}{\sqrt{\mathfrak{A}}}$, $\nu' = \frac{\mathfrak{G}}{\sqrt{\mathfrak{A}}}$. The condition of contact becomes

$$\frac{\mathrm{K}}{\sqrt{\mathfrak{A}}}\lambda \pm \mathrm{K} = 0.$$

And taking the under sign, $\lambda = \sqrt{\mathfrak{A}}$, so that if in the above written equation we establish all or any of the equations $\lambda = \sqrt{\mathfrak{A}}$, $\mu = \sqrt{\mathfrak{B}}$, $\nu = \sqrt{\mathfrak{C}}$, we have the equation of a section touching all or the corresponding sections of the sections

$$x=0,\ y=0,\ z=0.$$

In particular we have for a solution of the problem of tactions, the following equation of the section touching $x=0$, $y=0$, $z=0$, viz.

$$(a\sqrt{\mathfrak{A}} + h\sqrt{\mathfrak{B}} + g\sqrt{\mathfrak{C}})x + (h\sqrt{\mathfrak{A}} + b\sqrt{\mathfrak{B}} + f\sqrt{\mathfrak{C}})y + (g\sqrt{\mathfrak{A}} + f\sqrt{\mathfrak{B}} + c\sqrt{\mathfrak{C}})z$$
$$+ \frac{\sqrt{-p}}{\sqrt{\mathrm{K}}}\sqrt{2(\sqrt{\mathfrak{B}\mathfrak{C}} - \mathfrak{F})(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})}\, w = 0.$$

Anticipating the use of a notation the value of which will subsequently appear, or putting

$$\mathrm{f} = \sqrt[4]{\mathfrak{A}}\sqrt{\sqrt{\mathfrak{B}\mathfrak{C}} - \mathfrak{F}},\quad \mathrm{g} = \sqrt[4]{\mathfrak{B}}\sqrt{\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G}},\quad \mathrm{h} = \sqrt[4]{\mathfrak{C}}\sqrt{\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H}},\quad \mathrm{J} = \sqrt{2}\sqrt[4]{\mathfrak{A}\mathfrak{B}\mathfrak{C}},$$

values which give

$$\mathrm{K}^2 = -\mathrm{f}^4 - \mathrm{g}^4 - \mathrm{h}^4 + 2\mathrm{g}^2\mathrm{h}^2 + 2\mathrm{h}^2\mathrm{f}^2 + 2\mathrm{f}^2\mathrm{g}^2 - \frac{4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2},$$

the equation of the section in question is

$$\frac{\mathrm{f}^2}{\sqrt{\mathfrak{A}}}(-\mathrm{f}^2 + \mathrm{g}^2 + \mathrm{h}^2)x + \frac{\mathrm{g}^2}{\sqrt{\mathfrak{B}}}(\mathrm{f}^2 - \mathrm{g}^2 + \mathrm{h}^2)y + \frac{\mathrm{h}^2}{\sqrt{\mathfrak{C}}}(\mathrm{f}^2 + \mathrm{g}^2 - \mathrm{h}^2)z + \frac{\mathrm{fgh}\sqrt{-p}\sqrt{\mathrm{K}}}{\mathrm{J}}w = 0.$$

I proceed to investigate a transformation of the equation for the section with an indeterminate parameter λ, which touches the two sections $y=0$, $z=0$. We have

$$a\nabla^2=(a\lambda+h\mu+g\nu)^2+(\mathfrak{C}\mu^2+\mathfrak{B}\nu^2-2\mathfrak{F}\mu\nu)-\mathfrak{B}\mathfrak{C}+\mathfrak{F}^2;$$

or putting for μ and ν their values $\sqrt{\mathfrak{B}}$, $\sqrt{\mathfrak{C}}$ in the second term,

$$a\nabla^2=(a\lambda+h\mu+g\nu)^2+(\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F})^2;$$

or introducing instead of λ an indeterminate quantity X, such that

$$a\lambda+h\mu+g\nu=(\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F})\mathrm{X},$$

we have

$$a\nabla^2=(\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F})\sqrt{1+\mathrm{X}^2}.$$

Also introducing throughout X instead of λ, and completing the substitution of $\sqrt{\mathfrak{B}}$, $\sqrt{\mathfrak{C}}$ for μ, ν, the equation of the section touching $y=0$, $z=0$, becomes

$$(ax+hy+gz)\mathrm{X}+y\sqrt{\mathfrak{C}}+z\sqrt{\mathfrak{B}}+\sqrt{-ap}\sqrt{1+\mathrm{X}^2}.w=0.$$

And it may be remarked in passing, that this is a very convenient form for the demonstration of the theorem; "If two sections of a surface of the second order touch each other, and are also tangent sections (of the same class) to two fixed sections, then considering the planes through the axis of the fixed sections and the poles of the tangent sections, and also the tangent planes through this axis, the anharmonic ratio of the four planes is independent of the position of the moveable tangent sections;" where by the axis of the fixed sections is to be understood the line joining their poles.

The sections touching $z=0$, $x=0$, and $x=0$, $y=0$, are of course

$$x\sqrt{\mathfrak{C}}+(hx+by+fz)\mathrm{Y}+z\sqrt{\mathfrak{A}}+\sqrt{-bp}\sqrt{1+\mathrm{Y}^2}.w=0$$

$$x\sqrt{\mathfrak{B}}+y\sqrt{\mathfrak{A}}+(gx+fy+cz)\mathrm{Z}+\sqrt{-cp}\sqrt{1+\mathrm{Z}^2}.w=0,$$

where

$$h\lambda'+b\mu'+f\nu'=(\sqrt{\mathfrak{C}\mathfrak{A}}-\mathfrak{G})\mathrm{Y},\quad \lambda'=\sqrt{\mathfrak{A}},\ \mu'=\mu',\ \nu'=\sqrt{\mathfrak{C}}$$

$$g\lambda''+f\mu''+c\nu''=(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\mathrm{Z},\quad \lambda''=\sqrt{\mathfrak{A}},\ \mu''=\sqrt{\mathfrak{B}},\ \nu''=\nu''.$$

The conditions of contact of the sections represented by the above written equations would be perhaps most simply obtained directly from the lemma, but it is proper to deduce it from the formula for contact used in the present memoir. If for shortness

$$\Phi(\pm)=a\lambda'\lambda''+b\mu'\mu''+c\nu'\nu''+f(\mu'\nu''+\mu''\nu')+g(\nu'\lambda''+\nu''\lambda')+h(\lambda'\mu''+\lambda''\mu')\pm\mathrm{K},$$

where the symbol Φ $(\pm)$ is used in order to mark the essentially different character of the results corresponding to the different values of the ambiguous sign, then

$$\begin{aligned}bc\Phi(-)=&f(h\lambda'+b\mu'+f\nu')(g\lambda''+f\mu''+c\nu''),\\&+(\mathfrak{A}\nu'-\mathfrak{G}\lambda')(g\lambda''+f\mu''+c\nu''),\\&+(\mathfrak{A}\mu''-\mathfrak{H}\lambda'')(h\lambda'+b\mu'+f\nu'),\\&+\nu'\mu''(-\mathfrak{A}f)+\nu'\lambda''f\mathfrak{H}+\lambda'\mu''f\mathfrak{G}+\lambda'\lambda''(\mathrm{K}-f\mathfrak{F})\\&-\mathfrak{A}\mathrm{K}-f^2\mathrm{K}.\end{aligned}$$

$$
\begin{aligned}
&= f(h\lambda'+b\mu'+f\nu')(g\lambda''+f\mu''+c\nu'') \\
&\qquad +\sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(g\lambda''+f\mu''+c\nu'') \\
&\qquad +\sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})(h\lambda'+b\mu'+f\nu') \\
&\qquad +f\left(-\mathfrak{A}\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{H}\sqrt{\mathfrak{C}\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{A}\mathfrak{F}-(\mathfrak{G}\mathfrak{H}-\mathfrak{A}\mathfrak{F})\right) \\
&= f(h\lambda'+b\mu'+f\nu')(g\lambda''+f\mu''+c\nu'') \\
&\qquad +\sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(g\lambda''+f\mu''+c\nu'') \\
&\qquad +\sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})(h\lambda'+b\mu'+f\nu') \\
&\qquad -f(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}),
\end{aligned}
$$

i. e.

$$bc\Phi(-)=(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\{f\mathrm{YZ}+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})-f\}.$$

What, however, is really required*, is the value of $\Phi(+)$; to find this,

$$
\begin{aligned}
bc\Phi(+)&=bc\Phi(-)+2bc\mathrm{K} \\
&=(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\{f\mathrm{YZ}+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+f\} \\
&\quad +2bc\mathrm{K}-2f(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}),
\end{aligned}
$$

the second line of which is

$$
\begin{aligned}
&2(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\left\{\frac{bc\mathrm{K}}{\mathrm{K}^2bc}(\sqrt{\mathfrak{A}\mathfrak{C}}+\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}+\mathfrak{H})-f\right\} \\
&\quad =\frac{2(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})}{\mathrm{K}}\{(\sqrt{\mathfrak{A}\mathfrak{C}}+\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}+\mathfrak{H})-\mathfrak{G}\mathfrak{H}+\mathfrak{A}\mathfrak{F}\} \\
&\quad =2(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\sqrt{\mathfrak{A}}\,\theta,
\end{aligned}
$$

where

$$\theta=\frac{1}{\mathrm{K}}(\sqrt{\mathfrak{A}\mathfrak{B}\mathfrak{C}}+\mathfrak{F}\sqrt{\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{B}}+\mathfrak{H}\sqrt{\mathfrak{C}})\,;$$

and consequently

$$bc\Phi(+)=(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\{f\mathrm{YZ}+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+f+2\theta\sqrt{\mathfrak{A}}\},$$

a reduction, which on account of its peculiarity, I have thought right to work out in full.

The condition of contact is

$$\Phi(+)=\nabla'\nabla''=\frac{1}{\sqrt{bc}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}.$$

* It may be shown without difficulty that the $(-)$ sign would imply that the sections touching $z=0$, $x=0$ and $x=0$, $y=0$ were sections touching $x=0$ at the same point. By taking the $(-)$ sign in each equation we should have the solution of the problem "to determine three sections of a surface of the second order, the two sections of each pair touching one of three given sections at the same point," which is not without interest; the solution may be completed without any difficulty.

Or finally, the condition in order that the sections

$$x\sqrt{\mathfrak{C}}+(hx+by+fz)\mathrm{Y}+z\sqrt{\mathfrak{A}}+\sqrt{-bp}\sqrt{1+\mathrm{Y}^2}w=0$$

$$x\sqrt{\mathfrak{B}}+y\sqrt{\mathfrak{A}}+(gx+fy+cz)\mathrm{Z}+\sqrt{-cp}\sqrt{1+\mathrm{Z}^2}w=0$$

(the former of which is a section touching $z=0$, $x=0$, and the latter a section touching $x=0$, $y=0$) may touch, is

$$f\mathrm{YZ}+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+(f+2\theta\sqrt{\mathfrak{A}})-\sqrt{bc}\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=0.$$

The preceding researches show that the solution of STEINER's extension of MALFATTI's problem depends on a system of equations, such as the system mentioned at the commencement of the following section.

§ 5.

Consider the system of equations

$$\alpha+\beta(\mathrm{Y}+\mathrm{Z})+\gamma\mathrm{YZ}+\delta\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=0$$

$$\alpha'+\beta'(\mathrm{Z}+\mathrm{X})+\gamma'\mathrm{ZX}+\delta'\sqrt{1+\mathrm{Z}^2}\sqrt{1+\mathrm{X}^2}=0$$

$$\alpha''+\beta''(\mathrm{X}+\mathrm{Y})+\gamma''\mathrm{XY}+\delta''\sqrt{1+\mathrm{X}^2}\sqrt{1+\mathrm{Y}^2}=0.$$

These equations may, it will be seen, be solved by quadratics only, when the coefficients satisfy the relations

$$\frac{\beta}{\gamma-\alpha}=\frac{\beta'}{\gamma'-\alpha'}=\frac{\beta''}{\gamma''-\alpha''}$$

$$\frac{\beta^2+\gamma^2-\delta^2}{\gamma^2-\alpha^2}=\frac{\beta'^2+\gamma'^2-\delta'^2}{\gamma'^2-\alpha'^2}=\frac{\beta''^2+\gamma''^2-\delta''^2}{\gamma''^2-\alpha''^2},$$

equations which it should be remarked are satisfied by

$$\beta=0,\ \beta'=0,\ \beta''=0,\ \gamma=\delta,\ \gamma'=\delta',\ \gamma''=\delta''.$$

Or if we write

$$\frac{\alpha}{\gamma}=-l,\ \frac{\alpha'}{\gamma'}=-m,\ \frac{\alpha''}{\gamma''}=-n,$$

the equations become by a simple reduction,

$$\mathrm{Y}^2+\mathrm{Z}^2+2l\mathrm{YZ}=l^2-1$$

$$\mathrm{Z}^2+\mathrm{X}^2+2m\mathrm{ZX}=m^2-1$$

$$\mathrm{X}^2+\mathrm{Y}^2+2n\mathrm{XY}=n^2-1,$$

which are equivalent to the equations discussed in my paper "On a System of Equations connected with MALFATTI's Problem and on another Algebraical System," Cambridge and Dublin Mathematical Journal, t. iv. p. 270; the solution might have been effected by the direct method, which I shall here adopt, of eliminating any one of the variables between the two equations into which it enters, and combining the result with the third equation.

Writing the second and third equations under the form

$$A'+B'X+C'\sqrt{1+X^2}=0$$

$$A''+B''X+C''\sqrt{1+X^2}=0,$$

the result of the elimination may be presented in the form

$$A'A''+B'B''-C'C''=\sqrt{A'^2+B'^2-C'^2}\sqrt{A''^2+B''^2-C''^2},$$

which is most easily obtained by writing $X=\tan\phi$ and operating with the symbol $\cos^{-1}$; but if the rationalized equations be represented by

$$\lambda'+2\mu'X+\nu'X^2=0 \text{ and } \lambda''+2\mu''X+\nu''X^2=0,$$

the form

$$4(\lambda'\nu'-\mu'^2)(\lambda''\nu''-\mu''^2)=(\lambda'\nu''+\lambda''\nu'-2\mu'\mu'')^2$$

leads easily to the result in question. The values which enter are

$$\begin{array}{ll} A'=\alpha'+\beta'Z & A''=\alpha''+\beta''Y \\ B'=\beta'+\gamma'Z & B''=\beta''+\gamma''Y \\ C'=\delta'\sqrt{1+Z^2} & C''=\delta''\sqrt{1+Y^2}; \end{array}$$

whence, in the first place, by the equation connecting Y, Z,

$$C'C''=-\frac{\delta'\delta''}{\delta}\{\alpha+\beta(Y+Z)+\delta YZ\}.$$

It is obviously convenient that $A'A''+B'B''$ should be symmetrical with respect to Y and Z, and this will be the case if

$$\alpha'\beta''+\beta'\gamma''=\alpha''\beta'+\beta''\gamma', \quad \textit{i.e.} \text{ if } \beta'(\gamma''-\alpha'')=\beta''(\gamma'-\alpha').$$

Or assuming that the equations are symmetrically related to the system, we have the first set of relations between the coefficients, relations which are satisfied by

$$\alpha=\gamma+2\phi\beta,\ \alpha'=\gamma'+2\phi\beta',\ \alpha''=\gamma''+2\phi\beta'',$$

and the values of α, α', α'' will be considered henceforth as given by these conditions. We have

$$A'A''+B'B''-C'C''=\alpha'\alpha''+\beta'\beta''+(\gamma'\beta''+\gamma''\beta'+2\phi\beta'\beta'')(Y+Z)+(\beta'\beta''+\gamma'\gamma'')YZ$$
$$+\frac{\delta'\delta''}{\delta}\{\alpha+\beta(Y+Z)+\gamma YZ\}.$$

The quantities $A'^2+B'^2-C'^2$, $A''^2+B''^2-C''$ are quadratic functions of Z and Y respectively, and with proper relations between the coefficients, we may assume

$$(A'^2+B'^2-C'^2)(A''^2+B''^2-C''^2)=l^2s^2\{U^2+k[(\alpha+\beta(Y+Z)+\gamma YZ)^2-\delta^2(1+Y^2)(1+Z^2)]\},$$

in which U is a linear function of $Y+Z$ and YZ, and k and ls are constants. The first side must, in the first place, be symmetrical with respect to Y and Z, or

$$\alpha'^2+\beta'^2-\delta'^2,\quad (\alpha'+\gamma')\beta',\quad \beta'^2+\gamma'^2-\delta'^2$$

must be proportional to

$$\alpha''^2+\beta''^2-\delta''^2,\quad (\alpha''+\gamma'')\delta'',\quad \beta''^2+\gamma''^2-\delta''^2.$$

But since

$$(\alpha'+\gamma')\beta',\ (\alpha''+\gamma'')\beta''$$

are proportional to

$$\gamma'^2-\alpha'^2,\ \gamma''^2-\alpha''^2,$$

it is only necessary that

$$\beta'^2+\gamma'^2-\delta'^2,\ \beta''^2+\gamma''^2-\delta''$$

should be proportional to

$$\gamma'^2-\alpha'^2,\ \gamma''^2-\alpha''^2.$$

Or since the equations are supposed symmetrically related to the system, we must have the second set of relations between the coefficients. Suppose

$$\frac{\beta^2+\gamma^2-\delta^2}{\gamma^2-\alpha^2}=\frac{\beta'^2+\gamma'^2-\delta'^2}{\gamma'^2-\alpha'^2}=\frac{\beta''^2+\gamma''^2-\delta''^2}{\gamma''^2-\alpha''^2}=-\frac{s}{\phi},$$

then since

$$\gamma^2-\alpha^2=-4(\gamma+\phi\beta)\phi\beta,\ \&c.,$$

we have

$$\begin{aligned}\delta^2\ &=\beta^2\ +\gamma^2\ -4s(\gamma\ +\phi\beta)\beta\\ \delta'^2\ &=\beta'^2\ +\gamma'^2\ -4s(\gamma'\ +\phi\beta')\beta'\\ \delta''^2&=\beta''^2+\gamma''^2-4s(\gamma''+\phi\beta'')\beta'',\end{aligned}$$

and δ, δ', δ'' will be supposed henceforth to satisfy these equations.

We have next

$$\begin{aligned}A'^2\ +B'^2\ -C'^2&=4(\gamma'+\phi\beta')\beta'(s+\phi+Z+sZ^2)\\ A''^2+B''^2-C''^2&=4(\gamma''+\phi\beta'')\beta''(s+\phi+Y+sY^2),\end{aligned}$$

which may be simplified by writing

$$s=\frac{\mu-\phi}{1+\mu^2},\qquad \nu=\frac{1+\mu\phi}{\mu-\phi},$$

where μ, ν are to be considered as given functions of s and ϕ. These values give

$$\begin{aligned}A'^2\ +B'^2\ -C'^2\ &=4(\gamma'+\phi\beta'\ )\beta's(Z+\mu)(Z+\nu)\\ A''^2+B''^2-C''^2&=4(\gamma''+\phi\beta'')\beta''s\ (Y+\mu)(Y+\nu).\end{aligned}$$

Hence, putting for simplicity

$$l^2=4(\gamma'+\phi\beta')(\gamma''+\phi\beta'')\beta'\beta'',$$

we have

$$4(Z+\mu)(Z+\nu)(Y+\mu)(Y+\nu)=U^2+k[(\alpha+\beta(Y+Z)+\gamma YZ)^2-\delta^2(1+Y^2)(1+Z^2)].$$

And the two sides have next to be expressed in terms of $Y+Z$ and YZ.

If for symmetry we write

$$\xi=1,\ \eta=Y+Z,\ \zeta=YZ,$$

then

$$4(\mu^2\xi+\mu\eta+\zeta)(\nu^2\xi+\nu\eta+\zeta)+k\delta^2[(\xi-\zeta)^2+\eta^2]=U^2+k(\alpha\xi+\beta\eta+\gamma\zeta)^2.$$

And U is now to be considered a linear function of ξ, η, ζ.

The condition that the first side of the equation may divide into factors, gives an equation for determining k; since the condition is satisfied for $k=0$ and $k=\infty$, the

equation will be linear, and it is easily seen that the value is $k=\frac{1}{\delta^2}(\mu-\nu)^2$. In fact

$$4(\mu^2\xi+\mu\eta+\zeta)(\nu^2\xi+\nu\eta+\zeta)^2+(\mu-\nu)^2[(\xi-\zeta)^2+\eta^2]$$
$$=(2\mu\nu\xi+(\mu+\nu)\eta+2\zeta)^2+(\mu-\nu)^2(\xi+\zeta)^2.$$

Hence

$$\{2\mu\nu\xi+(\mu+\nu)\eta+2\zeta\}^2-\mathrm{U}^2=\frac{(\mu-\nu)^2}{\delta^2}\{(\alpha\xi+\beta\eta+\gamma\zeta)^2-\delta^2(\xi+\zeta)^2\}.$$

And we may assume

$$2\mu\nu\xi+(\mu+\nu)\eta+2\zeta+\mathrm{U}=\frac{\mu-\nu}{\delta}\Lambda\{(\alpha\xi+\beta\eta+\gamma\zeta)-\delta(\xi+\zeta)\}$$

$$2\mu\nu\xi+(\mu+\nu)\eta+2\zeta-\mathrm{U}=\frac{\mu-\nu}{\delta}\frac{1}{\Lambda}\{(\alpha\xi+\beta\eta+\gamma\zeta)+\delta(\xi+\zeta)\},$$

subject to its being shown that

$$4\mu\nu\xi+2(\mu+\nu)\eta+4\zeta=\frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)(\alpha\xi+\beta\eta+\gamma\zeta)-\delta\left(\Lambda-\frac{1}{\Lambda}\right)(\xi+\zeta)\right\}$$

gives a constant value for Λ. The comparison of coefficients gives

$$4\mu\nu=\frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)\alpha-\left(\Lambda-\frac{1}{\Lambda}\right)\delta\right\}$$

$$2\mu+2\nu=\frac{\mu-\nu}{\delta}\left(\Lambda+\frac{1}{\Lambda}\right)\beta$$

$$4=\frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)\gamma-\left(\Lambda-\frac{1}{\Lambda}\right)\delta\right\},$$

the first and third of which give

$$4(1-\mu\nu)=\frac{\mu-\nu}{\delta}\left(\Lambda+\frac{1}{\Lambda}\right)(\gamma-\alpha),$$

which will be identical with the second, if

$$\frac{2(1-\mu\nu)}{\mu+\nu}=\frac{\beta}{\gamma-\alpha}=-2\phi,$$

which follows at once from the equation

$$\nu=\frac{1+\mu\phi}{\mu-\phi}.$$

Forming next the two equations

$$\Lambda+\frac{1}{\Lambda}=\frac{2}{(\mu-\nu)\beta}(\mu+\nu)\delta$$

$$\Lambda-\frac{1}{\Lambda}=\frac{2}{(\mu-\nu)\beta}\{(\mu+\nu)\gamma-2\beta\}$$

these will be equivalent to a single equation if

$$(\mu+\nu)^2\delta^2=\{(\mu+\nu)\gamma-2\beta\}^2+(\mu-\nu)^2\beta^2,$$

i. e. if

$$(\mu+\nu)^2\delta^2=(\mu+\nu)^2(\beta^2+\gamma^2)-4(\mu+\nu)\beta\gamma-4(\mu\nu-1)\beta^2;$$

or finally, if

$$\delta^2=\beta^2+\gamma^2-4s\beta\left(\gamma+\frac{\mu\nu-1}{\mu+\nu}\beta\right)=\beta^2+\gamma^2-4s(\gamma+\phi\beta)\beta,$$

which is in fact the case.

Writing the equations for $\quad \Lambda+\frac{1}{\Lambda},\ \Lambda-\frac{1}{\Lambda}$

in the form

$$\Lambda+\frac{1}{\Lambda}=\frac{2\delta}{(\mu-\nu)\beta s}$$

$$\Lambda-\frac{1}{\Lambda}=\frac{2}{(\mu-\nu)\beta s}(\gamma-2\beta s),$$

and substituting in $\quad \mathrm{U}=\frac{\mu-\nu}{2\delta}\left\{\left(\Lambda-\frac{1}{\Lambda}\right)(\alpha\xi+\beta\eta+\gamma\zeta)-\left(\Lambda+\frac{1}{\Lambda}\right)\delta^2(\xi+\zeta)\right\},$

we have

$$\mathrm{U}=\frac{1}{s\beta\delta}\{(\gamma-2\beta s)(\alpha\xi+\beta\eta+\gamma\zeta)-\delta^2(\xi+\zeta)\}$$

$$=\frac{1}{s\beta\delta}\{(-\beta+2s\gamma+2\phi\gamma)\xi+(\gamma-2s\beta)\eta+(-\beta+2s\gamma+4s\phi\beta)\zeta\}.$$

And consequently, multiplying by

$$ls=2\sqrt{(\gamma'+\phi\beta')(\gamma''+\phi\beta'')\beta'\beta''}$$

we have

$$\sqrt{\mathrm{A}'^2+\mathrm{B}'^2-\mathrm{C}'^2}\sqrt{\mathrm{A}''^2+\mathrm{B}''^2-\mathrm{C}''^2}$$

$$=\frac{2}{\delta}\sqrt{(\gamma'+\phi\beta')(\gamma''+\phi\beta'')\beta'\beta''}\{(-\beta+2s\gamma+2\phi\gamma)\xi+(\gamma-2s\beta)\eta+(-\beta+2s\gamma+4s\phi\beta)\zeta\},$$

or collecting the different terms

$$(\alpha'\alpha''+\beta'\beta'')\xi+(\gamma'\beta''+\gamma''\beta'+2\phi\beta'\beta'')\eta+(\beta'\beta''+\gamma'\gamma'')\zeta+\frac{\delta'\delta''}{\delta}(\alpha\xi+\beta\eta+\gamma\zeta)$$

$$-\frac{2}{\delta}\sqrt{(\gamma'+2\phi\beta')(\gamma''+2\phi\beta'')\beta'\beta''}\{(-\beta+2s\gamma+2\phi\gamma)\xi+(\gamma-2s\beta)\eta+(-\beta+2s\gamma+4s\phi\beta)\zeta\}=0,$$

which, combined with the first equation written under the form

$$(\alpha\xi+\beta\eta+\gamma\zeta)^2-\delta^2[(\xi-\zeta)^2+\eta^2]=0,$$

determines the ratios of ξ, η, ζ, *i. e.* the values of $\mathrm{Y}+\mathrm{Z}$ and YZ.

§ 6.

The system of equations

$$(f+2\theta\sqrt{\mathfrak{A}})+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+f\mathrm{YZ}-\sqrt{bc}\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=0$$

$$(g+2\theta\sqrt{\mathfrak{B}})+\sqrt{\mathfrak{B}}(\mathrm{Z}+\mathrm{X})+g\mathrm{ZX}-\sqrt{ca}\sqrt{1+\mathrm{Z}^2}\sqrt{1+\mathrm{X}^2}=0$$

$$(h+2\theta\sqrt{\mathfrak{C}})+\sqrt{\mathfrak{C}}(\mathrm{X}+\mathrm{Y})+h\mathrm{XY}-\sqrt{ab}\sqrt{1+\mathrm{X}^2}\sqrt{1+\mathrm{Y}^2}=0,$$

where

$$\theta=\frac{1}{\mathrm{K}}(\sqrt{\mathfrak{ABC}}+\mathfrak{F}\sqrt{\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{B}}+\mathfrak{H}\sqrt{\mathfrak{C}}),$$

on which depends the solution of STEINER's extension of MALFATTI's problem, is at once seen to belong to the class of equations treated of in the preceding section, and

we have $\phi=\theta$, $s=0$. The equations at the conclusion of the preceding section become

$$\{\sqrt{\mathfrak{B}\mathfrak{C}}+gh+2\theta(g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}})+4\theta^2\sqrt{\mathfrak{B}\mathfrak{C}}\}\xi+\{g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}}+2\theta\sqrt{\mathfrak{B}\mathfrak{C}}\}\eta+\{\sqrt{\mathfrak{B}\mathfrak{C}}+gh\}\zeta$$

$$-a[(f+2\theta\sqrt{\mathfrak{A}})\xi+\sqrt{\mathfrak{A}}\eta+f\zeta]-\frac{2}{\sqrt{bc}}\sqrt{(g+\theta\sqrt{\mathfrak{B}})(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{B}\mathfrak{C}}}\{(\sqrt{\mathfrak{A}}-2\theta f)\xi-f\eta+\sqrt{\mathfrak{A}}\zeta\}=0$$

$$\{(f+2\theta\sqrt{\mathfrak{A}})\xi+\sqrt{\mathfrak{A}}\eta+f\zeta\}^2-bc\{(\xi-\zeta)^2+\eta^2\}=0,$$

which may also be written

$$(\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{F})(\xi+\zeta)+(-a\sqrt{\mathfrak{A}}+g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}}+2\theta\sqrt{\mathfrak{B}\mathfrak{C}})(\eta+2\theta\xi)$$

$$-\frac{2}{\sqrt{bc}}\sqrt{(g+\theta\sqrt{\mathfrak{B}})(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{B}\mathfrak{C}}}\big((\sqrt{\mathfrak{A}}-2\theta f)\xi-f\eta+\sqrt{\mathfrak{A}}\zeta\big)=0.$$

$$\{f(\xi+\zeta)+\sqrt{\mathfrak{A}}(\eta+2\theta\xi)\}^2-bc\{(\xi-\zeta)^2+\eta^2\}=0.$$

Or observing that

$$g+\theta\sqrt{\mathfrak{B}}=\frac{1}{\mathrm{K}}(\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{F})(\sqrt{\mathfrak{A}\mathfrak{B}}+\mathfrak{H})\,;\quad h+\theta\sqrt{\mathfrak{C}}=\frac{1}{\mathrm{K}}(\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{F})(\sqrt{\mathfrak{A}\mathfrak{C}}+\mathfrak{G})$$

$$-a\sqrt{\mathfrak{A}}+h\sqrt{\mathfrak{B}}+g\sqrt{\mathfrak{C}}+2\theta\sqrt{\mathfrak{B}\mathfrak{C}}=\theta(\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{F}),$$

and putting for a moment

$$\lambda=\frac{1}{\mathrm{K}}\sqrt{(\sqrt{\mathfrak{A}\mathfrak{C}}+\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}+\mathfrak{H})\sqrt{\mathfrak{B}\mathfrak{C}}},$$

and therefore

$$\sqrt{(g+\theta\sqrt{\mathfrak{B}})(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{B}\mathfrak{C}}}=(\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{F})\lambda,$$

the first equation divides by $(\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{F})$, and the result is

$$(\xi+\zeta)+\theta(\eta+2\theta\xi)-\frac{2\lambda}{\sqrt{bc}}\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}=0.$$

And by an easy transformation the second equation becomes

$$-\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}^2+4bc\zeta\big((\xi+\zeta)+\theta(\eta+2\theta\xi)-(1+\theta^2)\xi\big)=0.$$

Or putting

$$\xi+\zeta+\theta(\eta+2\theta\xi)=\Theta$$

$$\frac{1}{\sqrt{bc}}\big(\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\big)=\Phi$$

$$\zeta\qquad\qquad=\Psi,$$

the equations become

$$\Theta-2\lambda\Phi=0$$

$$-\Phi^2+4\Psi\{\Theta-(1+\theta^2)\Psi\}=0.$$

Whence eliminating Φ,

$$\left(2\Psi-\frac{\Theta}{1+\theta^2}\right)^2=\frac{\Theta^2}{(1+\theta^2)^2}\left(1-\frac{1+\theta^2}{4\lambda^2}\right),$$

or observing that

$$1+\theta^2=\frac{1}{\mathrm{K}^2}(\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{F})(\sqrt{\mathfrak{C}\mathfrak{A}}+\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}+\mathfrak{H}),$$

and reducing,

$$\Psi=\frac{K^2\Theta}{(\sqrt{\mathfrak{BC}}+\mathfrak{F})(\sqrt{\mathfrak{CA}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H})}\left(1+\frac{\sqrt{(\sqrt{\mathfrak{BC}}-\mathfrak{F})\sqrt{\mathfrak{BC}}}}{\sqrt{2\mathfrak{BC}}}\right).$$

Also $\Theta=2\lambda\Phi$ gives

$$\Phi=\frac{K\Theta}{2\sqrt{(\sqrt{\mathfrak{AC}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H})\sqrt{\mathfrak{BC}}}}.$$

Suppose

$$\sqrt{\mathfrak{BC}}+\mathfrak{F}=\alpha,\ \sqrt{\mathfrak{BC}}-\mathfrak{F}=\alpha_{\prime}\ \ \therefore\ \alpha\alpha_{\prime}=Ka$$

$$\sqrt{\mathfrak{CA}}+\mathfrak{G}=\beta,\ \sqrt{\mathfrak{CA}}-\mathfrak{G}=\beta_{\prime}\qquad \beta\beta_{\prime}=Kb$$

$$\sqrt{\mathfrak{AB}}+\mathfrak{H}=\gamma,\ \sqrt{\mathfrak{AB}}-\mathfrak{H}=\gamma_{\prime}\qquad \gamma\gamma_{\prime}=Kc;$$

then substituting

$$\Theta-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}\sqrt{bc}\Phi=0$$

$$\Theta-\frac{4bc}{\beta_{\prime}\gamma_{\prime}}(\alpha+\alpha_{\prime})\left(1-\frac{\sqrt{\alpha_{\prime}}}{\sqrt{\alpha+\alpha_{\prime}}}\right)\Psi=0,$$

that is,

$$\xi+\zeta+\theta(\eta+2\theta\xi)-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}=0$$

$$\xi+\zeta+\theta(\eta+2\theta\xi)-\frac{4bc(\alpha+\alpha_{\prime})}{\beta_{\prime}\gamma_{\prime}}\left(1-\frac{\sqrt{\alpha_{\prime}}}{\sqrt{\alpha+\alpha_{\prime}}}\right)\zeta=0,$$

which may be written

$$L\,\xi+M\,\eta+N\,\zeta=0$$

$$L'\xi+M'\eta+N'\zeta=0,$$

where

$$L=1+2\theta^2-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}(\sqrt{\mathfrak{A}}-2\theta f),\ M=\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}f,\ N=1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}\sqrt{\mathfrak{A}},$$

$$L'=1+2\theta^2-\frac{4bc(\alpha+\alpha_{\prime})}{\beta_{\prime}\gamma_{\prime}}\left(1-\frac{\sqrt{\alpha_{\prime}}}{\sqrt{\alpha+\alpha_{\prime}}}\right),\quad M'=\theta\qquad\qquad, N'=1.$$

Or since ξ, η, ζ are equal to 1, Y+Z, YZ respectively,

$$1:Y+Z:YZ=MN'-M'N:NL'-N'L:LM'-L'M$$

$$=-\frac{\sqrt{2}(\alpha+\alpha_{\prime})}{\sqrt{\beta_{\prime}\gamma_{\prime}}}(f+\theta\sqrt{\mathfrak{A}})$$

$$:\frac{4bc(\alpha+\alpha_{\prime})}{\beta_{\prime}\gamma_{\prime}}\left(1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}\sqrt{\mathfrak{A}}\right)\left(1-\frac{\sqrt{\alpha_{\prime}}}{\sqrt{\alpha+\alpha_{\prime}}}\right)+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}(f+\theta\sqrt{\mathfrak{A}})2\theta$$

$$:-\frac{4bc(\alpha+\alpha_{\prime})}{\beta_{\prime}\gamma_{\prime}}\left(\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}f\right)\left(1-\frac{\sqrt{\alpha_{\prime}}}{\sqrt{\alpha+\alpha_{\prime}}}\right)+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}(f+\theta\sqrt{\mathfrak{A}}).$$

Also

$$f+\theta\sqrt{\mathfrak{A}}=\frac{\beta\gamma}{K}=\frac{Kbc}{\beta_{\prime}\gamma_{\prime}},$$

whence

$$Y+Z=-\frac{2\sqrt{2}\sqrt{\alpha+\alpha_{,}}\sqrt{\beta_{,}\gamma_{,}}}{K}\left(1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\sqrt{\mathfrak{A}}\right)\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)-2\theta$$

$$YZ=\frac{2\sqrt{2}\sqrt{\alpha+\alpha_{,}}\sqrt{\beta_{,}\gamma_{,}}}{K}\left(\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}f}{\sqrt{\beta_{,}\gamma_{,}}}\right)\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)-1.$$

And by forming the analogous expressions for $Z+X$ and ZX, $X+Y$ and XY, the values of X, Y, Z may be determined. But the equations in question simplify themselves in a remarkable manner by the notation before alluded to.

Suppose

$$\mathrm{f}=\sqrt[4]{\mathfrak{A}}\sqrt{\sqrt{\mathfrak{BC}}-\mathfrak{F}},\ \mathrm{g}=\sqrt[4]{\mathfrak{B}}\sqrt{\sqrt{\mathfrak{CA}}-\mathfrak{G}},\ \mathrm{h}=\sqrt[4]{\mathfrak{C}}\sqrt{\sqrt{\mathfrak{AB}}-\mathfrak{H}},\ \mathrm{J}=\sqrt{2}\sqrt[4]{\mathfrak{ABC}},$$

these values give

$$\begin{aligned}
\frac{K\sqrt{\mathfrak{A}}}{\sqrt{\mathfrak{BC}}}a&=2\mathrm{f}^2\left(1-\frac{\mathrm{f}^2}{\mathrm{J}^2}\right)\\
\frac{K\sqrt{bc}}{\sqrt{\mathfrak{A}}}&=2\mathrm{gh}\sqrt{1-\frac{\mathrm{g}^2}{\mathrm{J}^2}}\sqrt{1-\frac{\mathrm{h}^2}{\mathrm{J}^2}}\\
\frac{Kf}{\sqrt{\mathfrak{A}}}&=\mathrm{f}^2-\mathrm{g}^2-\mathrm{h}^2+\frac{2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\\
&\vdots\\
K\theta&=-\mathrm{f}^2-\mathrm{g}^2-\mathrm{h}^2+2\mathrm{J}^2\\
K^2&=-\mathrm{f}^4-\mathrm{g}^4-\mathrm{h}^4+2\mathrm{g}^2\mathrm{h}^2+2\mathrm{h}^2\mathrm{f}^2+2\mathrm{f}^2\mathrm{g}^2-\frac{4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}.
\end{aligned}$$

Applying these results to the preceding formulæ and forming for that purpose the equations

$$2\sqrt{2}\sqrt{\alpha+\alpha_{,}}\sqrt{\beta_{,}\gamma_{,}}=4\mathrm{gh},\quad \frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}=\frac{\mathrm{J}^2}{\sqrt{\mathfrak{A}}\mathrm{gh}},\quad \frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}=\frac{\mathrm{f}}{\mathrm{J}}$$

$$\mathrm{gh}K\theta+\frac{\mathrm{J}^2}{\sqrt{\mathfrak{A}}}K\mathrm{f}=(\mathrm{J}^2-\mathrm{gh})\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)-2\mathrm{gh}(\mathrm{g}-\mathrm{h})^2,$$

we have

$$K(Y+Z)+2K\theta=4(\mathrm{J}^2-\mathrm{gh})\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)$$

$$K^2YZ+K^2=4\{(\mathrm{J}^2-\mathrm{gh})\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)-2\mathrm{gh}(\mathrm{g}-\mathrm{h})^2\}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right);$$

the former of which, combined with the similar equations for $Z+X$ and $X+Y$, gives for X, Y, Z the values to be presently stated, and these values will of course verify the second equation and the corresponding equations for ZX and XY.

Recapitulating the preceding notation, if $x=0$, $y=0$, $z=0$ are the equations of the given sections, $w=0$ the equation of the polar plane of their point of intersection with respect to the surface

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+pw^2=0,$$

the equation of the surface, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$, K as usual, and

$$\theta=\frac{1}{\mathrm{K}}(\sqrt{\mathfrak{ABC}}+\mathfrak{F}\sqrt{\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{B}}+\mathfrak{H}\sqrt{\mathfrak{C}}),$$

then the equations of the required sections are

$$\begin{aligned}&(ax+hy+gz)\mathrm{X}+y\sqrt{\mathfrak{C}}+z\sqrt{\mathfrak{B}}+\sqrt{-ap}\sqrt{1+\mathrm{X}^2}w=0\\&x\sqrt{\mathfrak{C}}+(hx+by+fz)\mathrm{Y}+z\sqrt{\mathfrak{A}}+\sqrt{-bp}\sqrt{1+\mathrm{Y}^2}w=0\\&x\sqrt{\mathfrak{B}}+y\sqrt{\mathfrak{A}}+(gx+fy+cz)\mathrm{Z}+\sqrt{-cp}\sqrt{1+\mathrm{Z}^2}w=0,\end{aligned}$$

where X, Y, Z are to be determined by the following equations,

$$\begin{aligned}&(f+2\theta\sqrt{\mathfrak{A}})+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+f\mathrm{YZ}-\sqrt{bc}\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=0\\&(g+2\theta\sqrt{\mathfrak{B}})+\sqrt{\mathfrak{B}}(\mathrm{Z}+\mathrm{X})+g\mathrm{ZX}-\sqrt{ca}\sqrt{1+\mathrm{Z}^2}\sqrt{1+\mathrm{X}^2}=0\\&(h+2\theta\sqrt{\mathfrak{C}})+\sqrt{\mathfrak{C}}(\mathrm{X}+\mathrm{Y})+h\mathrm{XY}-\sqrt{ab}\sqrt{1+\mathrm{X}^2}\sqrt{1+\mathrm{Y}^2}=0\,;\end{aligned}$$

and the solution of which, putting

$$\mathrm{f}=\sqrt[4]{\mathfrak{A}}\sqrt{\sqrt{\mathfrak{BC}}-\mathfrak{F}},\quad \mathrm{g}=\sqrt[4]{\mathfrak{B}}\sqrt{\sqrt{\mathfrak{CA}}-\mathfrak{G}},\quad \mathrm{h}=\sqrt[4]{\mathfrak{C}}\sqrt{\sqrt{\mathfrak{AB}}-\mathfrak{H}},\quad \mathrm{J}=\sqrt{2}\sqrt[4]{\mathfrak{ABC}},$$

is given by the equations

$$\begin{aligned}&\mathrm{KX}=\frac{2\mathrm{fgh}}{\mathrm{J}}+(-\mathrm{f}+\mathrm{g}+\mathrm{h})^2-2(-\mathrm{f}+\mathrm{g}+\mathrm{h})\mathrm{J}\\&\mathrm{KY}=\frac{2\mathrm{fgh}}{\mathrm{J}}+(\mathrm{f}-\mathrm{g}+\mathrm{h})^2-2(\mathrm{f}-\mathrm{g}+\mathrm{h})\mathrm{J}\\&\mathrm{KZ}=\frac{2\mathrm{fgh}}{\mathrm{J}}+(\mathrm{f}+\mathrm{g}-\mathrm{h})^2-2(\mathrm{f}+\mathrm{g}-\mathrm{h})\mathrm{J}\,*.\end{aligned}$$

Instead of the direct but very tedious process by which these values of X, Y, Z have been obtained, we may substitute the following *à posteriori* verification. We have

$$\begin{aligned}&\mathrm{K}^2(1+\mathrm{X}^2)=4(-\mathrm{f}+\mathrm{g}+\mathrm{h})^2\mathrm{J}^2\Big(1+\frac{\mathrm{f}}{\mathrm{J}}\Big)\Big(1-\frac{\mathrm{g}}{\mathrm{J}}\Big)\Big(1-\frac{\mathrm{h}}{\mathrm{J}}\Big)\\&\mathrm{K}^2\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=4\big(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\big)\mathrm{J}^2\Big(1-\frac{\mathrm{f}}{\mathrm{J}}\Big)\sqrt{1-\frac{\mathrm{g}^2}{\mathrm{J}^2}}\sqrt{1-\frac{\mathrm{h}^2}{\mathrm{J}^2}}\\&\mathrm{K}^2(1+\mathrm{YZ})=4\Big(1-\frac{\mathrm{f}}{\mathrm{J}}\Big)\Big\{(\mathrm{J}^2-\mathrm{gh})\big(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\big)-2\mathrm{gh}(\mathrm{g}-\mathrm{h})^2\Big\}\\&\mathrm{K}(\mathrm{Y}+\mathrm{Z})-2\mathrm{f}^2-\mathrm{g}^2-2\mathrm{h}^2+4\mathrm{J}^2=4\Big(1-\frac{\mathrm{f}}{\mathrm{J}}\Big)(\mathrm{J}^2-\mathrm{gh}).\end{aligned}$$

Putting also

$$\begin{aligned}&\mathrm{f}^2-\mathrm{g}^2-\mathrm{h}^2+\frac{2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}=\big(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\big)-\frac{2\mathrm{gh}(\mathrm{J}^2-\mathrm{gh})}{\mathrm{J}^2}\\&\mathrm{K}^2=\qquad\big(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\big)\Big((\mathrm{g}+\mathrm{h})^2-\mathrm{f}^2-\frac{4\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\Big)-\frac{4\mathrm{g}^2\mathrm{h}^2(\mathrm{g}-\mathrm{h})^2}{\mathrm{J}^2},\end{aligned}$$

* It is perhaps worth noticing that the value of the quantity λ previously made use of, is

$$\lambda=\frac{\mathrm{f}^2}{\mathrm{K}a\sqrt{\mathfrak{A}}}\Big\{\frac{2\mathrm{fgh}}{\mathrm{J}}-\mathrm{g}^2-\mathrm{h}^2+(\mathrm{J}+\mathrm{f}-\mathrm{g}-\mathrm{h})^2\Big\}.$$

we have

$$\left(\mathrm{f}^2-\mathrm{g}^2-\mathrm{h}^2+\frac{2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\right)\mathrm{K}^2(1+\mathrm{YZ})$$

$$=4\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)\left\{\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)\left[(\mathrm{J}^2-\mathrm{gh})\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)-2\mathrm{gh}(\mathrm{g}-\mathrm{h})^2-2\mathrm{gh}\frac{(\mathrm{J}^2-\mathrm{gh})^2}{\mathrm{J}^2}\right]\right.$$
$$\left.+\frac{4\mathrm{g}^2\mathrm{h}^2(\mathrm{g}-\mathrm{h})^2(\mathrm{J}^2-\mathrm{gh})}{\mathrm{J}^2}\right\}$$

$$\mathrm{K}^2\{\mathrm{K}(\mathrm{Y}+\mathrm{Z})-2\mathrm{f}^2-2\mathrm{g}^2-2\mathrm{h}^2+4\mathrm{J}^2\}$$

$$=4\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)\left\{\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)\left[(\mathrm{J}^2-\mathrm{gh})\left((\mathrm{g}+\mathrm{h})^2-\mathrm{f}^2-\frac{4\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\right)\right]-\frac{4\mathrm{g}^2\mathrm{h}^2(\mathrm{g}-\mathrm{h})^2(\mathrm{J}^2-\mathrm{gh})}{\mathrm{J}^2}\right\}.$$

Also, since

$$\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)+\left((\mathrm{g}+\mathrm{h})^2-\mathrm{f}^2-\frac{4\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\right)=4\mathrm{gh}\frac{(\mathrm{J}^2-\mathrm{gh})}{\mathrm{J}^2},$$

we have

$$\left(\mathrm{f}^2-\mathrm{g}^2-\mathrm{h}^2+\frac{2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\right)\mathrm{K}^2(1+\mathrm{YZ})+\mathrm{K}^2\{\mathrm{K}(\mathrm{Y}+\mathrm{Z})-2\mathrm{f}^2-2\mathrm{g}^2-2\mathrm{h}^2\}$$

$$=4\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)2\mathrm{gh}\mathrm{J}^2\left(1-\frac{\mathrm{g}^2}{\mathrm{J}^2}\right)\left(1-\frac{\mathrm{h}^2}{\mathrm{J}}\right).$$

And the values obtained above give also

$$2\mathrm{gh}\sqrt{1-\frac{\mathrm{g}^2}{\mathrm{J}^2}}\sqrt{1-\frac{\mathrm{h}^2}{\mathrm{J}^2}}\mathrm{K}^2\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}$$

$$=4\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)2\mathrm{gh}\mathrm{J}^2\left(1-\frac{\mathrm{g}^2}{\mathrm{J}^2}\right)\left(1-\frac{\mathrm{h}^2}{\mathrm{J}^2}\right),$$

which shows that the relation between Y and Z is verified by the assumed values of these quantities, and the other two equations are of course also verified. The solution of the problem will be rendered more complete if the equations of the required sections and of the auxiliary sections made use of in the geometrical construction are expressed in terms of f, g, h, J.

§ 7.

First, to substitute in the equations of the required sections or resultors. Writing the first equation in the form

$$\frac{\mathrm{K}^2}{2\sqrt{\mathfrak{B}\mathfrak{C}}}\left\{a\mathrm{X}x+(h\mathrm{X}+\sqrt{\mathfrak{C}})y+(g\mathrm{X}+\sqrt{\mathfrak{B}})z+\sqrt{-ap}\sqrt{1+\mathrm{X}^2}w\right\}=0,$$

the coefficient of x will be

$$\frac{\mathrm{f}^2}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}^2}{\mathrm{J}^2}\right)\left\{\frac{2\mathrm{fgh}}{\mathrm{J}}+(-\mathrm{f}+\mathrm{g}+\mathrm{h})^2-2\mathrm{J}(-\mathrm{f}+\mathrm{g}+\mathrm{h})\right\},$$

or, as it is convenient to write it,

$$\left(1+\frac{\mathrm{f}}{\mathrm{J}}\right)\mathrm{f}(-\mathrm{f}+\mathrm{g}+\mathrm{h})\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)\left\{\frac{2\mathrm{fgh}}{\mathrm{J}(-\mathrm{f}+\mathrm{g}+\mathrm{h})}-\mathrm{f}+\mathrm{g}+\mathrm{h}-2\mathrm{J}\right\}.$$

The coefficient of y is

$$\frac{1}{2\sqrt{\mathfrak{B}}}\left\{\left(-f^2-g^2+h^2+\frac{2g^2h^2}{J^2}\right)\left(\frac{2fgh}{J}+(-f+g+h)^2-2J(-f+g+h)\right)\right.$$

$$\left.-f^4-g^4-h^4+2g^2h^2+2h^2f^2+2f^2g^2-\frac{4f^2g^2h^2}{J^2}\right\},$$

or, after all reductions,

$$\left(1-\frac{f}{J}\right)f(-f+g+h)\frac{h}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)\left\{\frac{-2fgh}{J(-f+g+h)}+f-g+h+\frac{2J(f^2+g^2-h^2)}{2fg}\right\};$$

and similarly the coefficient of z is

$$\frac{1}{2\sqrt{\mathfrak{C}}}\left\{\left(-f^2+g^2-h^2+\frac{2h^2f^2}{J^2}\right)\left(\frac{2fgh}{J}+(-f+g+h)^2-2J(-f+g+h)\right)\right.$$

$$\left.-f^4-g^4-h^4+2g^2h^2+2h^2f^2+2f^2g^2-\frac{4f^2g^2h^2}{J^2}\right\},$$

or, after all reductions,

$$\left(1+\frac{f}{J}\right)f(-f+g+h)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)\left\{\frac{-2fgh}{J(-f+g+h)}+f+g-h+\frac{2J(f^2-g^2+h^2)}{2fh}\right\};$$

and the coefficient of w is

$$\left(1+\frac{f}{J}\right)f(-f+g+h)2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}.$$

Whence, forming the equation of the resultor in question, and by means of it those of the other resultors, the equations of the resultors are

$$\begin{aligned}&\left(\frac{2fgh}{J(-f+g+h)}-f+g+h-2J\right)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x\\
&+\left(\frac{-2fgh}{J(-f+g+h)}+f-g+h+2J\frac{f^2+g^2-h^2}{2fg}\right)\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y\\
&+\left(\frac{-2fgh}{J(-f+g+h)}+f+g-h+2J\frac{f^2-g^2+h^2}{2fh}\right)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z\\
&+2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}\,w=0\end{aligned}$$

$$\begin{aligned}&\left(\frac{-2fgh}{J(f-g+h)}-f+g+h+2J\frac{f^2+g^2-h^2}{2fg}\right)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x\\
&+\left(\frac{2fgh}{J(f-g+h)}+f-g+h-2J\right)\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y\\
&+\left(\frac{-2fgh}{J(f-g+h)}+f+g-h+2J\frac{-f^2+g^2+h^2}{2gh}\right)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z\\
&+2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}\,w=0\end{aligned}$$

$$\left(\frac{-2\mathrm{fgh}}{\mathrm{J}(\mathrm{f}+\mathrm{g}-\mathrm{h})}-f+g+h+2\mathrm{J}\frac{\mathrm{f}^2-\mathrm{g}^2+\mathrm{h}^2}{2\mathrm{fh}}\right)\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x$$
$$+\left(\frac{-2\mathrm{fgh}}{\mathrm{J}(\mathrm{f}+\mathrm{g}-\mathrm{h})}+\mathrm{f}-\mathrm{g}+\mathrm{h}+2\mathrm{J}\frac{-\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2}{2\mathrm{gh}}\right)\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y$$
$$+\left(\frac{2\mathrm{fgh}}{\mathrm{J}(\mathrm{f}+\mathrm{g}-\mathrm{h})}+\mathrm{f}+\mathrm{g}-\mathrm{h}-2\mathrm{J}\right)\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z$$
$$+2\sqrt{\mathrm{K}}\sqrt{1-\frac{\mathrm{f}}{\mathrm{J}}}\sqrt{1-\frac{\mathrm{g}}{\mathrm{J}}}\sqrt{1-\frac{\mathrm{h}}{\mathrm{J}}}\sqrt{-p}w=0,$$

values which might be somewhat simplified by writing ξ, η, ζ, ω instead of

$$\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x,\quad \frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y,\quad \frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z,\quad 2\sqrt{1-\frac{\mathrm{f}}{\mathrm{J}}}\sqrt{1-\frac{\mathrm{g}}{\mathrm{J}}}\sqrt{1-\frac{\mathrm{h}}{\mathrm{J}}}\sqrt{-p}w.$$

And it may be also remarked, that the coefficients as well of these formulæ as of those which follow may be elegantly expressed in terms of the parts of a triangle having f, g, h for its sides.

The equations of the separators are found by taking the differences two and two of the equations of the resultors (this requires to be verified *à posteriori*); thus subtracting the third equation from the second the result contains a constant factor,

$$\frac{1}{\mathrm{J}\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)\mathrm{gh}}\left\{4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2-\mathrm{J}\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)\left((\mathrm{g}+\mathrm{h})^2-\mathrm{f}^2\right)\right\},$$

equivalent to

$$\frac{1}{\mathrm{J}\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)\mathrm{gh}}\left(4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2-\mathrm{J}^2\left(\mathrm{K}^2+\frac{4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\right)\right)\quad\text{or}\quad\frac{-\mathrm{JK}^2}{\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)\mathrm{gh}}.$$

Rejecting the factor in question and forming the analogous two equations, the equations of the separators are

$$-\frac{\mathrm{g}-\mathrm{h}}{\mathrm{f}}\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x+\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y-\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z=0$$
$$-\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x-\frac{\mathrm{h}-\mathrm{f}}{\mathrm{g}}\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y+\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z=0$$
$$\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x-\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y-\frac{\mathrm{f}-\mathrm{g}}{\mathrm{h}}\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z=0\,;$$

and from the mode of formation of these equations it is obvious that the separators have a line in common.

The equations of the determinators being $x=0$, $y=0$, $z=0$, the equations of the tactors are

$$\sqrt{\mathfrak{B}}z-\sqrt{\mathfrak{C}}y=0,\quad \sqrt{\mathfrak{C}}x-\sqrt{\mathfrak{A}}z=0,\quad \sqrt{\mathfrak{A}}y-\sqrt{\mathfrak{B}}x=0\,;$$

and if $\alpha x+\beta y+\gamma z+\delta w=0$ be the equation of the tactor touching

$$x=0,\ \sqrt{\mathfrak{C}}x-\sqrt{\mathfrak{A}}z=0\ \text{and}\ \sqrt{\mathfrak{A}}y-\sqrt{\mathfrak{B}}x=0,$$

the conditions of contact are

$$\mathfrak{A}\left(\mathfrak{A}\alpha^2+\ldots\frac{\mathrm{K}}{p}\delta^2\right)=(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)^2,$$

$$2\sqrt{\mathfrak{AB}}(\sqrt{\mathfrak{AB}}-\mathfrak{H})\left(\mathfrak{A}\alpha^2+\ldots\frac{\mathrm{K}}{p}\delta^2\right)=\left\{(\sqrt{\mathfrak{AB}}-\mathfrak{H})(\alpha\sqrt{\mathfrak{A}}-\beta\sqrt{\mathfrak{B}})+\gamma(\mathfrak{G}\sqrt{\mathfrak{B}}-\mathfrak{H}\sqrt{\mathfrak{C}})\right\}^2,$$

$$2\sqrt{\mathfrak{AC}}(\sqrt{\mathfrak{AC}}-\mathfrak{G})\left(\mathfrak{A}\alpha^2+\ldots\frac{\mathrm{K}}{p}\delta^2\right)=\left\{(\sqrt{\mathfrak{AC}}-\mathfrak{G})(\alpha\sqrt{\mathfrak{A}}-\gamma\sqrt{\mathfrak{C}})+\beta(\mathfrak{H}\sqrt{\mathfrak{C}}-\mathfrak{F}\sqrt{\mathfrak{A}})\right\}^2,$$

whence

$$\frac{1}{\sqrt{\mathfrak{A}}}\sqrt{2\sqrt{\mathfrak{AB}}(\sqrt{\mathfrak{AB}}-\mathfrak{H})}(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)=$$
$$(\sqrt{\mathfrak{AB}}-\mathfrak{H})\sqrt{\mathfrak{A}}\alpha-(\sqrt{\mathfrak{AB}}-\mathfrak{H})\sqrt{\mathfrak{B}}\beta+(\mathfrak{G}\sqrt{\mathfrak{B}}-\mathfrak{F}\sqrt{\mathfrak{A}})\gamma$$

$$\frac{1}{\sqrt{\mathfrak{A}}}\sqrt{2\sqrt{\mathfrak{AC}}(\sqrt{\mathfrak{AC}}-\mathfrak{G})}(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)=$$
$$(\sqrt{\mathfrak{AC}}-\mathfrak{G})\sqrt{\mathfrak{A}}\alpha+(\mathfrak{H}\sqrt{\mathfrak{C}}-\mathfrak{F}\sqrt{\mathfrak{A}})\beta-(\sqrt{\mathfrak{AC}}-\mathfrak{G})\gamma$$

$$c\beta^2+b\gamma^2-2f\beta\gamma+\frac{\mathfrak{A}}{p}\delta^2=0,$$

and putting for a moment

$$\mu=\sqrt{\mathfrak{AC}}-\mathfrak{G}-\sqrt{2\sqrt{\mathfrak{AC}}(\sqrt{\mathfrak{AC}}-\mathfrak{G})}$$
$$\nu=\sqrt{\mathfrak{AB}}-\mathfrak{H}-\sqrt{2\sqrt{\mathfrak{AB}}(\sqrt{\mathfrak{AB}}-\mathfrak{H})}.$$

After some reductions, and observing that the ratios only of the quantities α, β, γ, δ are material,

$$\alpha=\frac{\mathrm{K}}{\sqrt{\mathfrak{A}}}(\mathrm{K}+h\nu+g\mu)$$
$$\beta=\frac{\mathrm{K}}{\sqrt{\mathfrak{A}}}(\qquad b\nu+f\mu)$$
$$\gamma=\frac{\mathrm{K}}{\sqrt{\mathfrak{A}}}(\qquad f\nu+c\mu)$$
$$\delta=\frac{\mathrm{K}}{\sqrt{\mathfrak{A}}}\sqrt{-p(b\nu^2+c\mu^2+2f\mu\nu)}\ ;$$

and it is easily seen also that the coordinates of the point of contact are

$$x=0,\quad y=\nu,\quad z=\mu,\quad w=-\frac{\delta}{p}\frac{\sqrt{\mathfrak{A}}}{\mathrm{K}}\ ;$$

also

$$\mu=-\frac{\mathrm{Jg}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right),\quad \nu=-\frac{\mathrm{Jh}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right).$$

And substituting and introducing throughout the quantities f, g, h, J, also forming the analogous equations, the equations of the tactors are

$$\left\{\mathrm{f}^2(-\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2)+(\mathrm{g}+\mathrm{h})\mathrm{J}\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2-\frac{2\mathrm{f}^2\mathrm{gh}}{\mathrm{J}^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{A}}}x$$
$$-\left\{\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2+\frac{2\mathrm{gh}(\mathrm{g}-\mathrm{h})}{\mathrm{J}}\right\}\mathrm{J}\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y$$
$$-\left\{\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2-\frac{2\mathrm{gh}(\mathrm{g}-\mathrm{h})}{\mathrm{J}}\right\}\mathrm{J}\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z$$
$$+2\sqrt{\mathrm{K}}\sqrt{\mathrm{gh}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)}\sqrt{-p}w=0$$

$$-\left\{g^2-(h-f)^2+\frac{2hf(h-f)}{J}\right\}J\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$

$$+\left\{g^2(f^2-g^2+h^2)+(h+f)J\left(g^2-(h-f)^2-\frac{2fg^2h}{J^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{B}}}y$$

$$-\left\{g^2-(h-f)^2-\frac{2hf(h-f)}{J}\right\}J\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z$$

$$+2\sqrt{K}\sqrt{hf\left(1-\frac{h}{J}\right)\left(1-\frac{f}{J}\right)\left(g^2-(h-f)^2\right)}\sqrt{-p}w=0$$

$$-\left\{h^2-(f-g)^2+\frac{2fg(f-g)}{J}\right\}J\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$

$$-\left\{h^2-(f-g)^2+\frac{2fg(f-g)}{J}\right\}J\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y$$

$$+\left\{h^2(f^2+g^2-h^2)+(f+g)J\left(h^2-(f-g)^2-\frac{2fgh^2}{J^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{C}}}z$$

$$+2\sqrt{K}\sqrt{fg\left(1-\frac{f}{J}\right)\left(1-\frac{g}{J}\right)\left(h^2-(f-g)^2\right)}\sqrt{-p}w=0.$$

It is obvious, from the equations, that each separator passes through the point of contact of a tactor and determinator, it consequently only remains to be shown that each separator touches two tactors. Consider the tactor which has been represented by $\alpha x+\beta y+\gamma z+\delta w=0$, the unreduced values of the coefficients give

$$\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma=K^2\sqrt{\mathfrak{A}}$$

$$\mathfrak{H}\alpha+\mathfrak{B}\beta+\mathfrak{F}\gamma=\frac{K^2}{\sqrt{\mathfrak{A}}}(\mathfrak{H}+\nu)$$

$$\mathfrak{G}\alpha+\mathfrak{F}\beta+\mathfrak{C}\gamma=\frac{K^2}{\sqrt{\mathfrak{A}}}(\mathfrak{G}+\mu)$$

$$\sqrt{\mathfrak{A}\alpha^2+\ldots\frac{K}{p}\delta^2}=\frac{1}{\sqrt{\mathfrak{A}}}(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)=K^2.$$

Represent for a moment the separator

$$\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x-\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y-\frac{f-g}{h}\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z=0$$

by $lx+my+nz+sw=0$. Then putting $\mathfrak{A}l^2+\ldots\frac{K}{p}s^2=\square^2$, since

$$\mathfrak{A}\alpha l+\ldots+\frac{K}{p}\delta s=K^2\left\{l\sqrt{\mathfrak{A}}+\frac{m}{\sqrt{\mathfrak{A}}}(\mathfrak{H}+\nu)+\frac{n}{\sqrt{\mathfrak{A}}}(\mathfrak{G}+\mu)\right\}$$

$$=K^2\left\{f\left(1-\frac{f}{J}\right)-g\left(1-\frac{2h}{J}\right)\left(1-\frac{g}{J}\right)-(f-g)\left(1-\frac{2g}{J}\right)\left(1-\frac{h}{J}\right)\right\}$$

$$=\frac{K^2}{J}\left\{-(f-g)^2+h(f+g)-\frac{2fgh}{J}\right\},$$

the condition of contact becomes

$$\square=\frac{1}{J}\left\{-(f-g)^2+h(f+g)-\frac{2fgh}{J}\right\}.$$

Or, forming the value of $\square^2$ and substituting

$$\mathrm{f}^2\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)^2+\mathrm{g}^2\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)^2+(\mathrm{f}-\mathrm{g})^2\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)^2$$

$$+2\left(1-\frac{2\mathrm{f}^2}{\mathrm{J}^2}\right)(\mathrm{f}-\mathrm{g})\mathrm{g}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)-2\left(1-\frac{2\mathrm{g}^2}{\mathrm{J}^2}\right)(\mathrm{f}-\mathrm{g})\mathrm{f}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)\left(1-\frac{\mathrm{f}}{\mathrm{g}}\right)-2\left(1-\frac{2\mathrm{h}^2}{\mathrm{J}^2}\right)\mathrm{fg}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)$$

$$=\frac{1}{\mathrm{J}^2}\left(-(\mathrm{f}-\mathrm{g})^2+\mathrm{h}(\mathrm{f}+\mathrm{g})-\frac{2\mathrm{fgh}}{\mathrm{J}}\right)^2,$$

which may be verified without difficulty, and thus the construction for the resultors is shown to be true.

§ 8.

Several of the formulæ of the preceding sections of this memoir apply to any number of variables. Consider the surface (*i. e.* hypersurface)

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy\ldots+pt^2=0,$$

and the section (*i. e.* hypersection)

$$(a\lambda+h\mu+g\nu\ldots)x+(h\lambda+b\mu+f\nu\ldots)y+(g\lambda+f\mu+c\nu\ldots)z\ldots+\sqrt{-p}\,\nabla t=0,$$

where

$$\nabla^2=a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu\ldots-\mathrm{K},$$

the condition of contact with any other section represented by a similar equation is

$$a\lambda\lambda'+b\mu\mu'+c\nu\nu'+f(\mu\nu'+\mu'\nu)+g(\nu\lambda'+\nu'\lambda)+h(\lambda\mu'+\lambda'\mu)\ldots\pm\mathrm{K}=\nabla\nabla',$$

where K is the determinant formed with the coefficients $a, b, c, f, g, h, \ldots$ And consequently, by establishing all or any of the equations $\lambda=\sqrt{\mathfrak{A}}$, $\mu=\sqrt{\mathfrak{B}}$, $\nu=\sqrt{\mathfrak{C}}$, ... we have the condition in order that the section in question may touch all or the corresponding sections of the sections $x=0$, $y=0$, $z=0$, ...

Let $\mathfrak{n}$ be the number of the variables x, y, z.., then $\mathrm{K}^{\mathfrak{n}-1}=\left|\begin{array}{cccc}\mathfrak{A} & \mathfrak{H} & \mathfrak{G} & .. \\ \mathfrak{H} & \mathfrak{B} & \mathfrak{F} & \\ \mathfrak{G} & \mathfrak{F} & \mathfrak{C} & \\ \vdots & & & \end{array}\right|$

also
$$\mathrm{K}^{\mathfrak{n}-2}\{(a\lambda+h\mu+g\nu..)x+(h\lambda+b\mu+f\nu..)y+(g\lambda+f\mu+c\nu..)z\ldots\}$$

$$=-\left|\begin{array}{ccccc} & x & y & z & .. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} & \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} & \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} & \\ \vdots & & & & \end{array}\right|$$

whence also

$$\mathrm{K}^{\mathfrak{n}-2}(\nabla^2+\mathrm{K})=-\left|\begin{array}{ccccc} & \lambda & \mu & \nu & .. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} & \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} & \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} & \\ \vdots & & & & \end{array}\right| \quad \text{or } \mathrm{K}^{\mathfrak{n}-2}\nabla^2=-\left|\begin{array}{ccccc} 1 & \lambda & \mu & \nu & .. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} & \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} & \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} & \\ \vdots & & & & \end{array}\right|$$

and the equation of the section in question becomes

$$-\left|\begin{array}{cccc} & x & y & z\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\left|\begin{array}{cccc} 1 & \lambda & \mu & \nu\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\; t=0$$

also the condition of contact with the corresponding section is

$$-\left|\begin{array}{cccc} \mp 1 & \lambda & \mu & \nu\,.. \\ \lambda' & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu' & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu' & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| = \sqrt{-\left|\begin{array}{cccc} 1 & \lambda & \mu & \nu\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\sqrt{-\left|\begin{array}{cccc} 1 & \lambda' & \mu' & \nu'\,.. \\ \lambda' & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu' & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu' & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}$$

In particular the equation of the sections which touches all the sections $x=0$, $y=0$, $z=0$ .., is

$$-\left|\begin{array}{cccc} & x & y & z\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\left|\begin{array}{cccc} 1 & \sqrt{\mathfrak{A}} & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\; t=0$$

Again, the equations of the section touching $y=0$, $z=0$, ... and the section touching $x=0$, $z=0$, ... are

$$-\left|\begin{array}{cccc} & x & y & z\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\left|\begin{array}{cccc} 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\; t=0$$

$$-\left|\begin{array}{cccc} & x & y & z\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\left|\begin{array}{cccc} 1 & \sqrt{\mathfrak{A}} & \mu & \sqrt{\mathfrak{C}}\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\; t=0$$

and the condition of contact of these two sections is

$$-\left|\begin{array}{cccc} \mp 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| = \sqrt{-\left|\begin{array}{cccc} 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\sqrt{-\left|\begin{array}{cccc} 1 & \sqrt{\mathfrak{A}} & \mu & \sqrt{\mathfrak{C}}\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}$$

It would seem from the appearance of these equations that there should be some simpler method of obtaining the solution than the method employed in the previous part of this memoir.

2 *Stone Buildings*,
April 1852.